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Multipartite entangled state of continuum variables generated by an optical network

Hong-yi Fan[1,2,3] **and Tong-qiang Song**[4]

[1] CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
[2] Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, People's Republic of China
[3] Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China
[4] Department of Physics, Ningbo University, Ningbo, Zhejiang, People's Republic of China

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Abstract
We construct a kind of new multipartite entangled states of continuum variables, which are related to unitary group $U(n)$. Using the technique of integral within an ordered product of operators we prove that such states make up a complete representation in multimode Fock space. The new state can be generated by an optical network whose operation on an incoming photon distributes the photon among the outputs according to the unitary group transform. The potential use of the new state in quantum teleportation is briefly discussed.

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1. Introduction

In recent years entangled states have been applied to quantum computation, quantum teleportation, quantum cryptography and quantum superdense coding [1–3]. In an entangled quantum state, measurement performed on one part of the system provides information on the remaining part, as first pointed out by Einstein, Podolsky and Rosen (EPR) [4] in their famous paper arguing the incompleteness of quantum mechanics. EPR revealed the quantum entanglement by considering nonlocal correlations between two particles due to the zero-commutator between their relative position and total momentum, i.e., $[X_1 - X_2, P_1 + P_2] = 0$. Recently, applications of quantum entanglement involved in entangled states of continuous-variable have attracted much attention from physicists [5–8]. Quantum teleportation of arbitrary coherent states has been realized experimentally with bipartite entanglement built from two single-mode squeezed vacuum states combined at a beam splitter [9]. Enlightened by EPR, the common eigenvector $|\eta\rangle$ of $X_1 - X_2$ and $P_1 + P_2$ in two-mode Fock space was explicitly constructed [10, 11], it is

$$|\eta\rangle = \exp\left[-\tfrac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_2^\dagger a_1^\dagger\right]|00\rangle \qquad (1)$$

where $\eta = \frac{1}{\sqrt{2}}(\eta_1 + \mathrm{i}\eta_2)$ is a complex number, $|00\rangle$ is the two-mode vacuum state, $\left(a_i, a_i^\dagger\right)$, $i = 1, 2$, are two-mode Bose annihilation and creation operators in Fock space related to (X_i, P_i) by

$$X_i = \frac{1}{\sqrt{2}}\left(a_i + a_i^\dagger\right) \qquad P_i = \frac{1}{\sqrt{2}\mathrm{i}}\left(a_i - a_i^\dagger\right). \tag{2}$$

The $|\eta\rangle$ state obeys the eigenvector equations

$$\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta|\eta\rangle \qquad \left(a_2 - a_1^\dagger\right)|\eta\rangle = -\eta^*|\eta\rangle. \tag{3}$$

It then follows from (2) and (3) that

$$(X_1 - X_2)|\eta\rangle = \eta_1|\eta\rangle \qquad (P_1 + P_2)|\eta\rangle = \eta_2|\eta\rangle. \tag{4}$$

Using the technique of integral within an ordered product (IWOP) of operators we have proved that $|\eta\rangle$ satisfies the completeness relation

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1 \qquad \mathrm{d}^2\eta \equiv \frac{1}{2}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \tag{5}$$

and possesses the orthonormal property

$$\langle\eta'|\eta\rangle = \pi\delta(\eta - \eta')\,\delta(\eta^* - \eta'^*). \tag{6}$$

The Schmidt decomposition of $|\eta\rangle$ in momentum eigenstate $|p\rangle_i$ basis is

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}p|p + \eta_2\rangle_1 \otimes |-p\rangle_2\,\mathrm{e}^{-\mathrm{i}p\eta_1} \qquad P_i|p\rangle_i = p|p\rangle_i \qquad i = 1, 2. \tag{7}$$

One may directly use the $|\eta\rangle$ state to discuss quantum teleportation [12], entanglement swapping [13] and quantum dense coding [14]. One can also directly use $|\eta\rangle$ to reveal the correlative amplitude–operational phase entanglement [15], where the operational phase operator was introduced by the Mandel group in discussing an eight-port homodyne detector for phase measurement [16]. Besides, it is remarkable that the well-known two-mode squeezing operator [17]

$$\exp\left[f\left(a_1^\dagger a_2^\dagger - a_1a_2\right)\right] \equiv S_2 \tag{8}$$

has a natural and simple representation in the $|\eta\rangle$ basis [18], i.e.,

$$S_2 = \frac{1}{\mu}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta/\mu\rangle\langle\eta| \qquad \mu = \exp f \tag{9}$$

no wonder that the two-mode squeezed state itself is an entangled state which entangles the idle mode and signal mode as an outcome of a parametric-down conversion process in the frequency domain [16]. It is encouraging that the ideal entangled state $|\eta\rangle$ can be constructed by using a beam splitter. From [5–8] we know that the symmetric 50:50 beam splitter operates on a pair of incoming modes: one is the zero-momentum eigenstate $|p = 0\rangle_1$ and the other is the zero-position eigenstate $|x = 0\rangle_2$, the outgoing state is a bipartite entangled state, i.e.,

$$\exp\left[\pi\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)/4\right]|p = 0\rangle_1 \otimes |x = 0\rangle_2 = \exp\left[a_1^\dagger a_2^\dagger\right]|00\rangle. \tag{10}$$

Then making a local oscillator displacement $D(\eta) = \exp\left[\eta a_1^\dagger - \eta^* a_1\right]$,

$$D(\eta)\exp\left[a_1^\dagger a_2^\dagger\right]|00\rangle = |\eta\rangle \tag{11}$$

the state $|\eta\rangle$ is obtained. Two interesting and important questions thus naturally arise: 1. How to theoretically construct an n-mode entangled state of continuum variables, which is in form as simple as possible and is qualified to make up a new quantum mechanical representation? 2. How to implement these new states experimentally? To our knowledge, an explicitly general form of multipartite entangled states of continuum variables in n-mode Fock space, as a proper quantum mechanical representation, has not been reported in the literature earlier. To answer these questions, in section 2 we first briefly review some properties of optical networks related to the U_n group (unitary group). Then in section 3 we shall construct n-mode entangled states in Fock space explicitly which make full use of the property of the U_n group. We also derive the completeness relation of these states and calculate their inner product. Section 4 is devoted to taking a tripartite entangled state as an example. In section 5 we discuss how to use an optical multiport network to realize such a multimode entangled state. In section 6 we briefly mention a protocol for teleporting the 3-mode continuous entangled state.

2. Brief review of properties of some optical devices

From (10) and (11) we learn that to realize an entangled state we need a selected unitary state transform and to understand which physical system effects the desired operation on incoming states. It is well known that the basic operation of passive optical devices, such as beam-splitters, mirrors, optical fibres, mixers, lenses, phase shifter and interferometer, based on quantum optics components, play the role of transforming a pre-assigned set of states into another set. They are the tools of quantum coding and communication [19]. It is also well known that photon correlation experiments are usually demonstrated by the linear multiport (of which a beam splitter is the simplest). A multiport distributes any incoming photons with a definite probability into the outputs. For example, Zeilinger and his collaborators [20] considered a totally symmetric $2n$-port to be a device that distributes a photon entering an arbitrary input equally among the outputs. Thus we briefly review some properties of a linear optical device. Let a quantum system be represented by a linear combination of single-photon states

$$|\psi(0)\rangle_j = \sum_i^{n-1} f_i a_i^\dagger(0)|0, 0, \cdots, 0\rangle \tag{12}$$

where f_i is a set of numbers. Transferring the state (12) through a linear optical device, the outgoing state is obtained by the application of a unitary transform U, whose role is

$$g_i = \sum_{j=0}^{n-1} u_{ij} f_j. \tag{13}$$

The conservation of photon probability distribution demands

$$\sum_{j=0}^{n-1} |g_i|^2 = \sum_{j=0}^{n-1} |f_i|^2 \tag{14}$$

which implies that the optical transfer matrix u_{ij} must be a representation of the unitary group element, possessing the unimodular property

$$\sum_{j=0}^{n-1} u_{ij} u_{kj}^* = \delta_{ik}. \tag{15}$$

Correspondingly, the photon creation operator undergoes the transform

$$U(t) a_i^\dagger(0) U^{-1}(t) = \sum_j u_{ij}(t)\, a_j^\dagger(0) = a_i^\dagger(t) \tag{16}$$

$$U(t)|\psi(0)\rangle_j = \sum_i f_i a_i^\dagger(t)|0, 0, \ldots, 0\rangle = |\psi(t)\rangle_j \tag{17}$$

where the vacuum state is invariant under the U transform, and

$$U(t) = \sum_j |\psi(t)\rangle_{jj}\langle\psi(0)| = \mathrm{e}^{-\mathrm{i}Ht}. \tag{18}$$

where H is the Hamiltonian describing the optical device. $U(t)$ is provided by those optical devices which play such basic operations. It is then of interest to inquire which types of mode interactions are necessary to obtain a desired optical transfer matrix. In [21–24] some prescriptions for obtaining Hamiltonian operator for n-port linear optical networks are presented. In the following we shall use a unitary group related linear multiport to realize n-mode entangled states of continuous variables.

3. New *n*-mode entangled states representation

The new n-mode entangled state we introduce is

$$|\vec{\zeta}\rangle_u = \exp\left\{\sum_{i=1}^{n-1}\left[-\frac{1}{2}|\zeta_i|^2 + a_n^\dagger u_{ji}\left(a_i^\dagger - \zeta_i^*\right) + \zeta_i a_i^\dagger\right]\right\}|\vec{0}\rangle \tag{19}$$

where $\vec{\zeta}$ is an $(n-1)$-dimensional complex vector, u_{ji} is an element of unitary group $U(n-1)$, the subscript u of $|\vec{\zeta}\rangle_u$ denotes that the state is unitary group dependent, $|\vec{0}\rangle$ is the n-mode vacuum state, from (15) we know

$$\sum_{i=1}^{n-1} u_{ji}u_{ji}^* = \delta_{jj} = 1. \tag{20}$$

The form of $|\vec{\zeta}\rangle_u$ seems not very complicated. Obviously, when $n = 2, i = j = 1, u_{11} = 1$, (19) reduces to the bipartite entangled state as in (2). Using the normally ordered form of the vacuum state projector

$$|\vec{0}\rangle\langle\vec{0}| =: \exp\left[-\sum_{j=1}^{n} a_j^\dagger a_j\right]: \tag{21}$$

and the technique of integral within an ordered product (IWOP) of operators [25, 26] we can prove the completeness relation (because in (19) the subscript j is not a summed index, we can omit it in $|\vec{\zeta}\rangle_u$ in our later calculations),

$$\begin{aligned}\int\prod_i^{n-1}\frac{\mathrm{d}^2\zeta}{\pi}|\vec{\zeta}\rangle_{uu}\langle\vec{\zeta}| &= \int\prod_i^{n-1}\frac{\mathrm{d}^2\zeta}{\pi} : \exp\left\{\sum_{i=1}^{n-1}\left[-|\zeta_i|^2 + a_n^\dagger u_i\left(a_i^\dagger - \zeta_i^*\right)\right.\right.\\ &\quad \left.\left. + \zeta_i a_i^\dagger + (a_i - \zeta_i)\,u_i^* a_n + \zeta_i^* a_i - a_i^\dagger a_i\right] - a_n^\dagger a_n\right\}:\\ &= : \exp\left\{\sum_{i=1}^{n-1}\left[\left(a_i - a_n^\dagger u_i\right)\left(a_i^\dagger - u_i^* a_n\right) + a_n^\dagger u_i a_i^\dagger + a_i u_i^* a_n - a_i^\dagger a_i\right] - a_n^\dagger a_n\right\}:\\ &= : \exp\left\{\sum_{i=1}^{n-1} a_n^\dagger u_i u_i^* a_n - a_n^\dagger a_n\right\} := 1\end{aligned} \tag{22}$$

where in the last step we have used (20). Note that although $|\vec{\zeta}\rangle_u$ is a n-mode state, the integral is $n-1$ fold. Further, by introducing the n-mode Glauber–Klauder coherent state [27, 28]

$$|\vec{z}\rangle \equiv |z_1, z_2, \ldots, z_n\rangle = \exp\left[\sum_{i=1}^{n}\left(-\frac{1}{2}|z_i|^2 + z_i a_i^\dagger\right)\right]|\vec{0}\rangle \tag{23}$$

we know

$$\langle \vec{z}|\vec{\zeta}\rangle_u = \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}|z_i|^2 + \sum_{i=1}^{n-1}\left[-\frac{1}{2}|\zeta_i|^2 + z_n^* u_i(z_i^* - \zeta_i^*) + \zeta_i z_i^*\right]\right\} \tag{24}$$

and

$$_u\langle \vec{\zeta}'|\vec{z}\rangle = \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}|z_i|^2 + \sum_{i=1}^{n-1}\left[-\frac{1}{2}|\zeta_i'|^2 + (z_i - \zeta_i')u_i'^* z_n + \zeta_i'^* z_i\right]\right\}. \tag{25}$$

By virtue of the over-completeness relation

$$\prod_{i=1}^{n}\frac{\mathrm{d}^2 z_i}{\pi}|\vec{z}\rangle\langle\vec{z}| = 1 \tag{26}$$

we calculate the overlap of the new entangled states

$$\begin{aligned}
_{u'}\langle\vec{\zeta}'|\vec{\zeta}\rangle_u &= _{u'}\left\langle \vec{\zeta}'\int\prod_{i}^{n}\frac{\mathrm{d}^2 z_i}{\pi}|z_i, z_n\rangle\langle z_i, z_n|\vec{\zeta}\right\rangle_u \\
&= f\int\prod_{i}^{n-1}\left[\frac{\mathrm{d}^2 z_i}{\pi}\right]\int\frac{\mathrm{d}^2 z_n}{\pi}\exp\left\{-\sum_{i=1}^{n}|z_i|^2 + \sum_{i=1}^{n-1}[z_n^* u_i(z_i^* - \zeta_i^*)\right. \\
&\quad \left. + \zeta_i z_i^* + (z_i - \zeta_i')\, u_i'^* z_n + \zeta_i'^* z_i]\right\} \\
&= f\int\prod_{i}^{n-1}\left[\frac{\mathrm{d}^2 z_i}{\pi}\right]\exp\left\{-\sum_{i=1}^{n-1}|z_i|^2 + \sum_{i=1}^{n-1}\sum_{k=1}^{n-1}(z_i^* - \zeta_i^*)u_i u_k'^*(z_k - \zeta_k')\right. \\
&\quad \left. + \sum_{i=1}^{n-1}(\zeta_i z_i^* + \zeta_i'^* z_i)\right\} \\
&= f\int\prod_{i}^{n-1}\left[\frac{\mathrm{d}^2 z_i}{\pi}\right]\exp\left\{\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}[-z_i^*(\delta_{ik} - u_i u_k'^*)z_k\right. \\
&\quad \left. - z_i^* u_i u_k'^*\zeta_k' - \zeta_i^* u_i u_k'^* z_k + \zeta_i^* u_i u_k'^*\zeta_k' + \zeta_i\delta_{ik}z_k^* + \zeta_i'^*\delta_{ik}z_k]\right\} \\
&= f\int\prod_{i}^{n-1}\left[\frac{\mathrm{d}^2 z_i}{\pi}\right]\exp\{-z^*(I - v)\tilde{z} - z^* v\tilde{\zeta}' - \zeta^* v\tilde{z} + \zeta^* v\tilde{\zeta}' + \zeta\tilde{z}^* + \zeta'^*\tilde{z}\}
\end{aligned} \tag{27}$$

where $f \equiv \exp\left[-\frac{1}{2}\sum_{i=1}^{n-1}(|\zeta_i|^2 + |\zeta_i'|^2)\right]$, I is $(n-1)\times(n-1)$ unit matrix, for brevity in the last line of (27) we have set

$$z^* = (z_1, z_2, \ldots, z_{n-1})^* \qquad \zeta^* = (\zeta_1, \zeta_2, \ldots, \zeta_{n-1})^* \tag{28}$$

and we have defined

$$(v)_{ik} \equiv u_i u_k'^* = \begin{pmatrix} u_1 u_1'^* & u_1 u_2'^* & \cdots & u_1 u_{n-1}'^* \\ u_2 u_1'^* & u_2 u_2'^* & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ u_{n-1}u_1'^* & u_{n-1}u_2'^* & \cdots & u_{n-1}u_{n-1}'^* \end{pmatrix}. \tag{29}$$

Using the integral formula [29]

$$\int \prod_i^n \frac{\mathrm{d}^2 z_i}{\pi} \exp\left\{-\frac{1}{2}(z, z^*)\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \tilde{z} \\ \tilde{z}^* \end{pmatrix} + (\mu \nu^*)\begin{pmatrix} \tilde{z} \\ \tilde{z}^* \end{pmatrix}\right\}$$
$$= \left[\det\begin{pmatrix} C & D \\ A & B \end{pmatrix}\right]^{-1/2} \exp\left[\frac{1}{2}(\mu\nu^*)\begin{pmatrix} C & D \\ A & B \end{pmatrix}^{-1}\begin{pmatrix} \tilde{\nu}^* \\ \tilde{\mu} \end{pmatrix}\right] \tag{30}$$

we perform the integral in (27),

$$_{u'}\langle\vec{\zeta}'|\vec{\zeta}\rangle_u = f \int \prod_i^{N-1}\left[\frac{\mathrm{d}^2 z_i}{\pi}\right] \exp\left\{-\frac{1}{2}(z, z^*)\begin{pmatrix} 0 & I - \tilde{v} \\ I - v & 0 \end{pmatrix}\begin{pmatrix} \tilde{z} \\ \tilde{z}^* \end{pmatrix}\right.$$
$$\left. + (\zeta'^* - \zeta^* v, \zeta - \zeta'\tilde{v})\begin{pmatrix} \tilde{z} \\ \tilde{z}^* \end{pmatrix} + \zeta^* v \tilde{\zeta}'\right\}$$
$$= f\left[\det\begin{pmatrix} I - v & 0 \\ 0 & I - \tilde{v} \end{pmatrix}\right]^{-1/2} \exp\left\{\frac{1}{2}(\zeta'^* - \zeta^* v, \zeta - \zeta'\tilde{v})\right.$$
$$\left. \times \begin{pmatrix} I - v & 0 \\ 0 & I - \tilde{v} \end{pmatrix}^{-1}\begin{pmatrix} \tilde{\zeta} - v\tilde{\zeta}' \\ \widetilde{\zeta^* v - \zeta'^*} \end{pmatrix} + \zeta^* v \tilde{\zeta}'\right\}$$
$$= f[\det(I - v)]^{-1} \exp\left\{(\zeta'^* - \zeta^* v)\frac{1}{I - v}(\tilde{\zeta} - v\tilde{\zeta}') + \zeta^* v \tilde{\zeta}'\right\}. \tag{31}$$

4. A concrete example

For example, when $n = 3$, we take

$$\vec{\zeta} = (\eta, \sigma) \qquad \vec{\zeta}' = (\eta', \sigma') \qquad u_1 = \cos\theta \qquad u_2 = \sin\theta \qquad 0 < \theta \leqslant 2\pi \tag{32}$$

where η, σ, η' and σ' are all complex numbers, u_1 and u_2 are elements of the unitary group U_2. Then equation (19) becomes

$$|\vec{\zeta}\rangle_u \to |\eta, \sigma\rangle_\theta = \exp\left\{-\tfrac{1}{2}(|\eta|^2 + |\sigma|^2) + a_3^\dagger\left(a_1^\dagger - \eta^*\right)\cos\theta\right.$$
$$\left. + a_3^\dagger\left(a_2^\dagger - \sigma^*\right)\sin\theta + \eta a_1^\dagger + \sigma a_2^\dagger\right\}|000\rangle \tag{33}$$

which is a new tripartite entangled state. In this case, equation (31) reduces to

$$_{\theta'}\langle\eta', \sigma'|\eta, \sigma\rangle_\theta = \frac{1}{1 - \cos(\theta - \theta')} \exp\left\{\frac{1}{1 - \cos(\theta - \theta')}[\cos\theta'(\eta - \eta') + \sin\theta'(\sigma - \sigma')]\right.$$
$$\left. \times [\cos\theta(\eta'^* - \eta^*) + \sin\theta(\sigma'^* - \sigma^*)] + W\right\} \tag{34}$$

where $W \equiv \eta'^*\eta + \sigma'^*\sigma - \frac{1}{2}(|\eta|^2 + |\sigma|^2 + |\eta'|^2 + |\sigma'|^2)$, its detailed derivation is shown in the appendix. In particular, when $\theta' = \theta$, (34) becomes

$$_\theta\langle\eta', \sigma'|\eta, \sigma\rangle_\theta = \lim_{\epsilon \to 0} \exp\left\{-\frac{1}{\epsilon}|\cos\theta(\eta - \eta') + \sin\theta(\sigma - \sigma')|^2 + W\right\}$$
$$= \pi\delta[\cos\theta(\eta - \eta') + \sin\theta(\sigma - \sigma')]\,\delta\,[\cos\theta(\eta^* - \eta'^*) + \sin\theta(\sigma^* - \sigma'^*)]\,\mathrm{e}^W \tag{35}$$

where we have used the limiting formula of the δ-function

$$\lim_{\epsilon \to 0} \exp\left\{-\frac{1}{\epsilon}|\alpha|^2\right\} = \pi\delta(\alpha)\delta(\alpha^*). \tag{36}$$

Equation (35) indicates that $|\eta,\sigma\rangle_\theta$ is a partly orthonormal state. To see the entanglement involved in $|\eta,\sigma\rangle_\theta$ explicitly, we write $\eta = \eta_1 + \mathrm{i}\eta_2, \sigma = \sigma_1 + \mathrm{i}\sigma_2$ and make the Schmidt decomposition

$$|\eta,\sigma\rangle_\theta = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \mathrm{d}u\,\mathrm{d}v\, D(u,v)\,\mathrm{e}^{-\mathrm{i}u\eta_1-\mathrm{i}v\sigma_1}|(u+\eta_2)/\sqrt{2}\rangle_1$$
$$\otimes|(v+\sigma_2)/\sqrt{2}\rangle_2 \otimes |[(\sigma_2 - v)\sin\theta + (\eta_2 - u)\cos\theta]/\sqrt{2}\rangle_3 \tag{37}$$

where the three single-mode states all belong to the set of momentum eigenstates, and

$$D(u,v) = \exp\left\{-\frac{1}{4}[(\eta_2 - u)\sin\theta - (\sigma_2 - v)\cos\theta]^2\right\}. \tag{38}$$

5. Physical implementation of $|\vec{\zeta}\rangle_u$

In this section we discuss a protocol of implementing the n-mode entangled state $|\zeta\rangle_u$ experimentally with the use of an optical network. Supposing we have already a state $\exp\left[a_n^\dagger b^\dagger\right]|\vec{0}\rangle$, which entangles mode $a_n^\dagger$ and mode $b^\dagger$ (an ideal beam splitter operation applied to a momentum-squeezed vacuum state $\left(\text{mode}\, a_n^\dagger\right)$ and a position-squeezed vacuum state (mode $b^\dagger$) can yield such a state, see the explanation just before equation (10)). Letting the $b^\dagger$-mode photon entering an n-port optical network that distributes the photon among the outputs according to the unitary transform,

$$Ub^\dagger U^{-1} = \sum_{i=1}^{n-1} u_{ji}a_i^\dagger \qquad \sum_{i=1}^{n-1} u_{ji}u_{ji}^* = \delta_{jj} = 1 \tag{39}$$

thus the outgoing state together with the $a_n^\dagger$-mode is

$$U\exp\left[a_n^\dagger b^\dagger\right]|\vec{0}\rangle = \exp\left[a_n^\dagger\sum_{i=1}^{n-1}u_{ji}a_i^\dagger\right]|\vec{0}\rangle. \tag{40}$$

Then making a local oscillator displacement $\prod_i^{n-1} D_i(\zeta_i) = \prod_i^{n-1}\exp\left(\zeta_i a_i^\dagger - \zeta_i^* a_i\right)$ to effect the outgoing state, we obtain

$$\prod_i^{n-1} D_i(\zeta_i)\exp\left[a_n^\dagger\sum_{i=1}^{n-1}u_{ji}a_i^\dagger\right]|\vec{0}\rangle = \exp\left\{\sum_{i=1}^{n-1}\left[-\frac{1}{2}|\zeta_i|^2 + a_n^\dagger u_{ji}\left(a_i^\dagger - \zeta_i^*\right) + \zeta_i a_i^\dagger\right]\right\}|\vec{0}\rangle = |\vec{\zeta}\rangle_u. \tag{41}$$

Hence the ideal n-mode entangled state $|\vec{\zeta}\rangle_u$ can be realized.

6. Application of the new entangled states

In this section we briefly mention some applications of the new entangled states. Let us still take $|\eta\rangle$ and $|\eta,\sigma\rangle_\theta$ for example. From equations (33) and (22) we know

$$\int\frac{\mathrm{d}^2\eta\,\mathrm{d}^2\sigma}{\pi^2}|\eta,\sigma\rangle_{\theta\theta}\langle\eta,\sigma| = 1. \tag{42}$$

Thus any 3-mode state $|\,\rangle_{123}$ can be expanded as

$$|\,\rangle_{123} = \int\frac{\mathrm{d}^2\eta\,\mathrm{d}^2\sigma}{\pi^2}G(\eta,\sigma,\theta)|\eta,\sigma\rangle_{\theta 123} \tag{43}$$

where $G(\eta, \sigma, \theta)$ is the expansion coefficient

$$G(\eta, \sigma, \theta) = {}_{\theta}\langle \eta, \sigma|\rangle_{123}. \tag{44}$$

If Alice is able to teleport the 3-mode state $|\eta, \sigma\rangle_{\theta 123}$ to Bob, then she can teleport $|\rangle_{123}$ since it can be expanded by the set of $|\eta, \sigma\rangle_{\theta}$. Assume Alice shares a quantum channel composed of three bipartite entangled states, $|\eta_\alpha\rangle_{45} \otimes |\eta_\beta\rangle_{67} \otimes |\eta_\gamma\rangle_{89}$, with Bob, which means that Alice owns particles 4, 6 and 8, while Bob owns 5, 7 and 9. Then the total initial state is $|\eta, \sigma\rangle_{\theta 123} \otimes |\eta_\alpha\rangle_{45} \otimes |\eta_\beta\rangle_{67} \otimes |\eta_\gamma\rangle_{89}$. Alice makes a joint measurement denoted by $|\eta'\rangle_{1414}\langle\eta'| \otimes |\eta''\rangle_{2626}\langle\eta''| \otimes |\eta'''\rangle_{3838}\langle\eta'''|$ and then informs Bob of the data η', η'' and η''' via a classical channel. Then Bob can reconstruct the 3-mode entangled state $|\eta, \sigma\rangle_\theta$ in mode 579 by a suitable local unitary transform.

In summary, enlightened by the construction of the bipartite entangled state $|\eta\rangle$, we have constructed a kind of multipartite entangled state of continuum variables. The explicit form of this state in n-mode Fock space seems not very complicated, so it can be handled mathematically without much difficulty. Using the IWOP technique we have shown that such states $|\vec{\zeta}\rangle_u$ make up a complete representation in multimode Fock space. The $|\vec{\zeta}\rangle_u$ state can be generated by an optical network whose operation on an incoming photon distributes the photon among the outputs according to the unitary group transform. The potential use of the new state in quantum teleportation is discussed briefly.

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Appendix

To derive equation (34), we note that (29) and (32) yield

$$v = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta', \sin\theta') = \begin{pmatrix} \cos\theta\cos\theta' & \cos\theta\sin\theta' \\ \sin\theta\cos\theta' & \sin\theta\sin\theta' \end{pmatrix} \qquad vv = \cos(\theta - \theta')v \tag{45}$$

and

$$(I - v)^{-1} = G \begin{pmatrix} 1 - \sin\theta\sin\theta' & \cos\theta\sin\theta' \\ \cos\theta'\sin\theta & 1 - \cos\theta\cos\theta' \end{pmatrix} \qquad G \equiv \frac{1}{1 - \cos(\theta - \theta')}. \tag{46}$$

Due to

$$\begin{gathered} (\cos\theta', \sin\theta') \begin{pmatrix} 1 - \sin\theta\sin\theta' & \cos\theta\sin\theta' \\ \cos\theta'\sin\theta & 1 - \cos\theta\cos\theta' \end{pmatrix} = (\cos\theta', \sin\theta') \\ \begin{pmatrix} 1 - \sin\theta\sin\theta' & \cos\theta\sin\theta' \\ \cos\theta'\sin\theta & 1 - \cos\theta\cos\theta' \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \end{gathered} \tag{47}$$

we have

$$\begin{gathered} v(I - v)^{-1} = (I - v)^{-1}v = Gv \qquad v\frac{1}{I - v}v = Gvv = G\cos(\theta - \theta')v \\ \det(I - v) = G^{-1}. \end{gathered} \tag{48}$$

Hence equation (31) reduces to

$${}_{u'}\langle\vec{\zeta}'|\vec{\zeta}\rangle_u \to {}_{\theta}\langle\eta', \sigma'|\eta, \sigma\rangle_\theta = fG\exp\left\{G[\zeta^* v(\tilde{\zeta}' - \tilde{\zeta}) - \zeta'^* v\tilde{\zeta}'] + \zeta'^*\frac{1}{I_n - v}\tilde{\zeta}\right\}. \tag{49}$$

Making substitutions $f \to \exp\left[-\frac{1}{2}(|\eta|^2 + |\sigma|^2 + |\eta'|^2 + |\sigma'|^2)\right]$,

$$\zeta^* v(\tilde{\zeta}' - \tilde{\zeta}) \to \cos\theta\cos\theta'(\eta' - \eta)\eta^* + \cos\theta\sin\theta'(\sigma' - \sigma)\eta^* \\ + \sin\theta\cos\theta'(\eta' - \eta)\sigma^* + \sin\theta\sin\theta'(\sigma' - \sigma)\sigma^* \tag{50}$$

$$\zeta'^* v\tilde{\zeta}' \to \cos\theta\cos\theta'\eta'^*\eta' + \sin\theta\sin\theta'\sigma'^*\sigma' + \cos\theta\sin\theta'\eta'^*\sigma' + \sin\theta\cos\theta'\sigma'^*\eta' \tag{51}$$

and

$$\zeta'^* \frac{1}{I - v}\tilde{\zeta} \to G(\eta'^*, \sigma'^*) \begin{pmatrix} 1 - \sin\theta\sin\theta' & \cos\theta\sin\theta' \\ \cos\theta'\sin\theta & 1 - \cos\theta\cos\theta' \end{pmatrix} \begin{pmatrix} \eta \\ \sigma \end{pmatrix} \tag{52}$$

into (A5) we reach to (34).

References

[1] Ekert A and Josza R 1996 *Rev. Mod. Phys.* **68** 733
[2] Bennett C H *et al* 1993 *Phys. Rev. Lett.* **70** 1895
[3] Huches C A *et al* 1997 *Phys. Rev.* A **56** 1163 and references therein
[4] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[5] Bouwmeester D *et al* 1999 *Phys. Rev. Lett.* **82** 1345
[6] DiVincenzo D P 1995 *Science* **270** 255
[7] Hofmann H F, Ide T, Kobayashi T and Furusawa A 2000 *Phys. Rev.* A **62** 062 304
[8] Cochrane P T and Milburn G J 2001 *Phys. Rev.* A **64** 062 312 and references therein
[9] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
[10] Fan Hong-yi and Klauder J R 1994 *Phys. Rev.* A **49** 704
[11] Fan Hong-yi and Ye Xiong 1995 *Phys. Rev.* A **51** 3343
Fan Hong-yi 2002 *Phys. Lett.* A **303** 311
[12] Fan Hong-yi 2002 *Phys. Lett.* A **294** 253
[13] Fan Hong-yi 2001 *Phys. Lett.* A **286** 81
[14] Fan Hong-yi 2001 *Phys. Lett.* A **291** 61
[15] Fan Hong-yi 2002 *J. Phys. A: Math. Gen* **35** 1007
[16] See, for example, Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[17] Loudon R and Knight P L 1987 *J. Mod. Opt* **37** 709
[18] Fan Hongyi and Fan Yue 1996 *Phys. Rev.* A **54** 958
Fan Hong-yi 2002 *Phys. Rev.* A **65** 064 102
[19] Stenholm S 1994 *J. Mod. Opt.* **41** 2483
Stenholm S 1995 *Quantum Semiclass. Opt.* **7** 667
Ekert A K and Knight P L 1990 *Phys. Rev.* A **42** 487
Ekert A K and Knight P L 1991 *Phys. Rev.* A **43** 3934
[20] Törma P, Stenholm S and Jex J *Phys. Rev.* A **52** 4853
[21] Törma P and Stenholm S 1995 *J. Mod. Opt.* **42** 1109
[22] Reck M, Zeilinger A, Bernstein H and Bertani P 1994 *Phys. Rev. Lett.* **73** 58
[23] Jex J, Stenholm S and Zeilinger A 1995 *Opt. Commun.* **117** 95
[24] Fan Hong-yi and Min Xiao 1997 *Quantum Semiclass. Opt.* **9** 53
[25] Fan Hong-yi, Zaidi H R and Klauder J R 1987 *Phys. Rev.* D **35** 1831
Fan Hong-yi and Xu Zhi-hua 1994 *Phys. Rev.* A **50** 2921
[26] Fan Hong-yi and Chen Jun-hua 2001 *J. Phys. A: Math. Gen.* **34** 10 939
Fan Hong-yi and Vandelinde J 1991 *J. Phys. A: Math. Gen.* **24** 2529
[27] Glauber R J 1963 *Phys. Rev.* **130** 2529
Glauber R J 1963 *Phys. Rev.* **131** 2766
[28] Klauder J R and Skargerstam B S 1985 *Coherent States* (Singapore: World Scientific)
[29] Berezin F A 1966 *The Method of Second Quantization* (New York: Academic)